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ABSTRACT

The principles of cognitive psychology as they relate to rational number learning and instruction are discussed. Issues of knowledge acquisition, organization, retrieval and transfer are reviewed as they affect this area of the elementary school curriculum. Specific problems in learning about rational numbers and systematic student errors are described with psychological interpretations offered for them. The various representational models of rational number as partitioned geometric areas, number lines, and discrete sets of objects, used in instruction, are discussed in relation to the difficulty in learning and integrating complex rational number concepts and procedures. One suggestion offered is that teachers point out to students the ways in which each rational number interpretation is different from the others while also showing what attributes they share; otherwise students may not see the commonalities that exist. Procedures and concepts may exist in isolation as encapsulated bits of knowledge which rarely are accessed and soon fade from disuse. (JN)

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Rational Number Learning and Instruction from a Cognitive Perspective

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The central question of this paper concerns the relevance of psychological research for one area of the mathematics curriculum, rational numbers. In particular, can the models of human cognition as limited capacity information processing speak to the problems of teaching and learning about rational numbers? Survey studies, such as the National Assessment of Educational Progress (1973, 1978, 1982), have documented student difficulties within this area of the mathematics curriculum. In examining these results, math educators have described student performance on rational number tasks as the mechanical execution of rote procedures (Carpenter, T.P., Corbitt, M.K., Kepner, H.S., Lindquist, M.M., Reys, R.E., 1981; Post, 1981) These memorized routines, because they lack a conceptual structure to substantiate them, may be easily distorted, confused with other overlearned procedures, or even forgotten.

This paper will examine some recurring student errors and descriptions of student performance provided by mathematics educators in light of work in cognitive psychology. First the problem of learning rational number concepts and procedures will be discussed. Then, student errors will be described with possible psychological interpretations offered for them. Finally, typical instructional treatments of rational numbers will be considered with suggestions as to how they may help to alleviate some of the difficulties experienced by students in learning about rational numbers.

What Is the Problem?

Carpenter et al. (1981) and Post (1981), in discussing the National Assessment results, propose that student errors

illustrate a mechanistic view of matnematics as memorized procedures. For these students, there appears to be no conerent structure of concepts and procedures upon which to build an integrated system of mathematical knowledge. Instead there exists an ever increasirg set of rules for symbol manipulation, each associated with a particular problem type and accessed only when a narrowly defined set of conditions are met. The task of distinguishing between conditions and the associated rules becomes so burdensome that confusions are made with no conceptual basis for detecting or correcting them. The resulting knowledge structure is an inadequate foundation upon which to build future mathematical knowledge. Furthermore, concepts and procedures may exist in isolation from each other, making transfer improbable within the domain of mathematics and across domains (science and social studies, for example).

What are the origins of this mechanical approach to mathematics learning? While instruction may be aimed at the development of conceptual as well as procedural knowledge, it may be inappropriately matched to a student's developmental level or may not have included or highlighted information necessary for comprehension. In this case, students may choose to memorize what they cannot understand. Alternatively, instruction may encourage memorization by focusing on the acquisition of automated procedures that can be retrieved and executed without understanding.

The inability to understand what is being taught could have several sources. First students may not be able to reconcile what is to be learned with what they already know. The new information

just does not fit with anything they have seen before. Unable to make sense out of what is presented, they may be forced to memorize and hope they can retrieve whatever is necessary to accomplish the task (Posner, Strike, Hewson, & Gertzog, 1982). Another possibility is that rational number concepts and procedures have been presented in a way that does not allow students to extract meaning. The amount of information that must be processed simultaneously for comprehension may exceed their processing capacity (Case, 1980). For example, it may be very difficult for students to understand complicated procedures involving fractions (e.g. a/b + c/d) when they do not automatically process and access the meaning of the component symbols. The limited resource of attention must be focused on decoding the symbols, leaving a limited amount available for comprehension of the procedure (LaBerge, 1976). A third possibility is that students have ignored some features of the problem which are essential for comprehension because they were not salient or they did not seem to be important (Siegler, 1982). For any of these reasons, students may rely on memorization techniques as a strategy for learning.

Connections between conceptual and procedural knowledge may not be made because instruction has emphasized the efficient execution of symbolic algorithms, without comprehension of the principles supporting them. If students define the task as "getting an answer", and it can be produced by memorizing, with no apparent loss in accuracy, why not do so. For many reasons (lack of interest in mathematics or knowledge about fractions; limitations of time, experience, or management skills; beliefs

about learning; or reliance upon textbooks), teachers may spend most instructional time presenting and drilling on the use of mathematical procedures, with insufficient time allotted to establishing meaning for them or relating them to what has been learned in the past.

The processing alternatives for assigned instructional tasks must be considered. Are students required to analyze problems in terms of mathematical principles, or to execute memorized rules? Are students exposed to novel problem solving situations where it is necessary to extract the mathematical relationship described to choose appropriate procedures? Or, are so-called applications limited to a few verbal problems appended to a set of computation exercises, all using the same mathematical procedure? These questions must be answered if we are to understand some of the origins of problems mentioned above. They are researchable questions open for future inquiry.

A mechanistic approach to rational number learning may result because rational numbers are very complex and may take on different forms depending on the type of mathematical relationship to be represented. Additionally, rational number procedures are considerably more complicated than the whole number procedures children have learned in the past. Mathematically, rational numbers are defined as the set of numbers that can be written in the form a/b, where a and b are integers and $b \neq 0$. In their study of mathematics, students may encounter rational numbers expressed as fractions, decimals, quotients, ratios, units of measure, or multiplicative operators (see Kieren, 1976 and 1980, for a discussion of each interpretation of rational number). Some

of the difficulty in learning about and using rational numbers may stem from an inability to integrate the various interpretations of rational number; to see how they are related and yet distinct. Perhaps each is learned at a superficial level, making it very difficult, if not impossible, for meaningful connections to be made between them.

Cognitive psychologists (Freeland & Larkin, 1983) warn that memorized rules, without the supporting conceptual structure, are fragile and easily disrupted. They characterize expertise within a domain as the coordination of both automated procedures and "elaborated representations" (representations in terms of underlying principles versus surface characteristics of the problem), which are combined for efficient problem solving (Freeland & Larkin, 1983). Within the domain of mathematics then, expertise would include the representation of a given problem in terms of mathematical relationships, which leads to the retrieval and execution of automatic procedures to obtain a solution (Larkin, 1983).

Rational Number Errors

Student performance on rational number procedures seems to fall short of the "expert" characterization in many cases. First, relying on an impoverished knowledge base, students may function at a very superficial level, depending on the exact duplication of contextual cues to prompt retrieval of appropriate concepts and procedures and leading to inflexible performance. Secondly, procedures which have been learned or practiced less well may soon fade from memory or be distorted by and/or confused with more familiar ones.

While students' procedures are automatic, their retrieval is often based on surface features (e.g. the vocabulary used- "altogether" signals addition and "difference" signals subtraction), rather than an elaborated mathematical representation of the problem. Rules may be mechanically applied, without regard to mathematical principles or the rationality of what results. Errors can occur when the surface clues contradict the mathematical relationship underlying the problem (such that the wrong procedure is fired), or provide conflicting or insufficient information (such that procedures are fired at random), or the procedures themselves are incorrect (this is highly probable when one must depend entirely upon memory without any means to substantiate or confirm what is retrieved). In some cases, however, students' errors may reveal their efforts to understand and integrate what they have learned, thus allowing us to see what makes sense to them and, possibly, how we can bring them to a more complete understanding.

Evidence exists for procedural errors in comparing or operating upon fractions, demonstrating the susceptibility of these procedures to intrusions from other, overlearned, whole number (counting) procedures (Post, 1981; Behr, Post, & Wachsmuth, 1982). Prior to and during the first few years of elementary school, a highly practiced counting routine frequently is used in problem solving. (Evidence for the use of counting for whole number addition includes the Count-All, Count-On and Min Models described by Resnick and Ford, 1981, and others.) The counting procedure is so familiar that order relationships between whole numbers are easily obtained by retrieving their positions in the

counting sequence (Behr et al., 1982). Counting and one-to-one correspondence also is used by young children to share a set of objects equally among the recipients, avoiding use of a partitioning scheme (the part-whole fraction interpretation) (Hiebert and Tonnessen, 1978).

It is quite natural, therefore, for children to retrieve the same counting procedure, though most times inappropriately, when ordering fractions. As Behr et al. (1982) document, children explain that, "One-third is less than one-fourth because three is less than four" (p. 1). While this strategy does not yield the correct result here, it should be noted that the correct answer is obtained when applied to ordering fractions with like denominators (three-fourths is less than four-fourths because three is less than four). In fact, one text book series uses this technique to teach the ordering of fractions with like denominators, illustrated as follows:

$$3/4 < 4/4$$

$$3 < 4$$

A counting procedure, which has been used very effectively in the past to order numbers, now proves to be correct for only one subset of fractions (those with like denominators). Where one routine was operative for all numbers, now two routines, with different conditions for their retrieval and execution, are required. Anderson (1983) describes this form of "production tuning" as the discrimination of conditions for existing procedures when error feedback indicates that more specific productions must be formed to avoid mistakes. It is not surprising that the older, more practiced procedure may often

block retrieval of the new routine which is more complicated, may be less well learned and has been rehearsed less often.

Carpenter et al. (1981) point out that 13-year-olds and 17-year-olds make similar mistakes when asked to estimate the answer to problems like, 12/13 + 7/8. Over half of the 13-year-olds and over a third of the 17-year-olds added either the numerators or the denominators of the fractions. They seemed to search for rules which they thought would work without considering whether the answers produced were sensible.

Psychological Interpretations of These Errors

How might cognitive psychologists interpret the student errors described above? Research concerning the organization of knowledge in memory and its retrieval would appear to speak most directly to the issue, if our goal is for mathematical learning to result in the formation of a coherent knowledge structure, with meaningful associations between concepts and routines stored in memory. Alternate paths for retrieval could be formed and strengthened as common features of fractions, decimals, quotients, etc. were recognized, leading to flexibility in the type of procedures invoked.

Anderson (1983) describes declarative memory as a "tangled hierarchy" (hierarchies sharing subhierarchies), a network structure of interconnected cognitive units of multiple representational types (propositions, strings, or spatial images), or nodes, and connections among them, or links. Each node has an associated strength which varies as a function of its use. The more frequently a node is used, the greater will be its associated strength. Information which is highly elaborated (connected to

other pieces of information in meaningful ways) is accessed frequently (increasing node strength) and better remembered by providing alternate retrieval routes and bases for inferring what cannot be directly recalled (Anderson, 1980).

For rational numbers then, an elaborated network of concepts and automatized procedures for operating upon them would result from experiences which formed appropriate associative links between the interpretations of rational number (fractions, decimals, quotients, etc.) and between each interpretation and related operations (productions with conditions specified as mathematical principles or relations versus superficial features of the problem). Links and retrieval paths would be strengthened through repeated associations, as a function of experience. The errors described above may arise because there do not exist connections between rational number concepts interpreted as fractions, decimals, quotients, etc. and the symbolic algorithms for operating upon them (procedures are unattached to concrete representations, experiences, or images which give them meaning). Furthermore, students may compartmentalize knowledge about each interpretation and fail to recognize common attributes and procedures.

Rather than a unified dynamic system of rational number knowledge, each interpretation and related procedures may exist as an isolated microworld of knowledge (Lawler, 1981), unattached and inaccessible through other microworlds. In a more interactive system, a particular mathematical situation, e.g. the presence of decimals, would result in the activation of the decimal node (concepts, images and related procedures), with activation

spreading to other rational number nodes and varying in the extent that previous connections had been made between them.

The question arises as to when and how the formation of meaningful associations between rational number interpretations can be made. One vehicle might be the model used to represent and give meaning to each interpretation; another could be the symbol used to express the mathematical relationship involved; a third might be procedures shared by different interpretations. The extent to which overlap exists across interpretations and procedures could influence the strength and nature of connections made and the flexibility of performance in operating upon rational numbers.

Hiebert (1984) proposes three sites in the problem solving process at which connections between conceptual and procedural knowledge (understanding and form) can be made. At the first site, meaning is established for the numerical and operational symbols as referents for objects and actions. For example, students could attach meaning to the fraction symbol through manipulative and pictorial experiences with various models of the fraction concept.

At site two, links are made between the steps of an algorithm and a physical representation, so that the reasonableness of the procedure is established. Area models could be used to justify addition and subtraction operations for fractions.

Site three, Hiebert suggests, involves estimation training or validating the reasonableness of a solution. Students could be explicitly taught how to estimate the answer to a fraction computation problem prior to the introduction of the formal

algorithm, by relating their knowledge about the quantities and operations involved to the symbol manipulation procedures. For example, in multiplying by 1/2, finding "1/2 of", we are partitioning into two parts, the value of one of those parts signifying the answer. When dividing by 1/2, we are asking, "How many halves (1/2s) are there in the given quantity?" As estimation skills improve, students will be more able to judge the sensibleness of their computational results. The degree to which instructional treatments of rational number contribute to the formation of useful connections will be discussed below.

Instructional Treatments of Rational Number

There are several distinct textbook models (different physical representations) used to represent rational numbers. These models vary from continuous areas (geometric regions which are partitioned and shaded to indicate fractional parts) and lengths (number lines partitioned to indicate fractional units), to discrete set-subset or set-set models (sets of objects partitioned into fractional subsets), to symbolic algorithms (abstract procedures for symbol manipulation). Any particular model may be more effective for illustrating one specific interpretation of rational number or an operation upon it. While text books may not soley determine the instruction received, they are recognized as powerful influences on what is taught (Brophy, 1982).

Area (Region) Models

Students first encounter rational numbers as fractions, or part-whole relationships, which are modeled by partitioned continuous areas, such as:

1/2

1/3

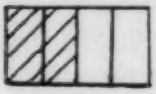
(See Pothier & Sawada, 1983, for a discussion of young children's partitioning and knowledge about fractions.) These models can provide meaning for the fraction symbol, as referents for actions (Hiebert, 1984; VanEngen, 1949), especially when students generate models of the fraction concept on their own, discussing each feature of the model and the corresponding attributes of fractions. Students should be thoroughly familiar with the concept of fraction before they are introduced to the fraction symbol. Expecting students to simultaneously attend to and learn the meaning of the fraction concept and its relationship to the fraction symbol may lead to cognitive overload, as discussed below.

In the symbol 1/2, the 2 represents the total number of equal parts into which the whole has been partitioned, while the 1 represents the number of those parts we are attending to at the moment. As pointed out earlier, teachers would need to provide sufficient experience, so that students come to process the symbol "a/b" as a unitized code (LaBerge, 1976), i.e. as a conceptual unit, rather than discrete symbols, "a", "/", "b", where each must be interpreted separately to decipher meaning. In accessing the conceptual unit "1/2", the student should also have access to the previous partitioning experiences, or images, which give it meaning.

Failure to encode "a/b" as a unit may explain the difficulty some students have in comprehending and learning rational number procedures. Students who must focus attention on decoding the meaning of the symbols "1/4" or "2/4" will have limited resources available for comprehending operational procedures involving them, such as "1/4 + 2/4". The mental effort used in decoding each symbol limits the amount available for learning new procedures.

Area models are also used to generate equivalent fractions and to illustrate addition and subtraction of fractions (establish connections between symbolic procedures and actions upon a concrete representation). Highlighting the equality of parts, these partitioned areas can be further subdivided to form equivalent fractions without changing the size of the unit, as illustrated below:

1/2

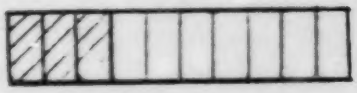
2/4 

As Greeno (1976) notes, this model of fraction equivalence corresponds to invariant quantity. Other models (the set-theoretic model) illustrate a different notion of equivalence to be discussed later.

Partitioned regions are useful for illustrating addition and subtraction of fractions, especially when common denominators must be generated for a solution (Coburn, 1973; Greeno, 1976). The reasonableness of addition and subtraction procedures can be established by mapping actions in the physical representation to

the steps of the algorithm (Hiebert, 1984; Resnick & Ford, 1981). These mappings should be conducted by students as well as teachers so that they form a rich store of concrete experiences (episodic memories) and images that can be used to validate the steps of an algorithm, to infer steps which may be forgotten, or to provide multiple access routes for retrieval of the correct procedure. Reliance upon a memorized procedure alone could lead to the problems described earlier.

Examination of several textbook series (Heath, 1983; Houghton Mifflin, 1978; Scott Foresman, 1975) indicates that area representations are used to introduce decimal notation as a special way to symbolize this type of fraction:



3/10 can be written 0.3

As students become comfortable with and make connections between the fraction symbol (3/10), the area model used and the corresponding decimal symbol (0.3), it may be easier for them to form other associations. For example, when the decimal symbol (0.1) is used to denote place value in the decimal system of numeration as well as a special kind of fraction, students may connect place value (the tenths place) with the fraction symbol (1/10) and the fraction concept (a tenth). Teachers will need to point out the links between the familiar fraction concept and its name (a tenth) and the concept of place value and the associated name (the tenths place). Such connections should be advantageous.

Number Line Models

Several researchers have shown that students and teachers

have difficulty using the number line model for fractions (Behr, et al., 1984; Larson, 1980; Leinhardt, 1983). The problem seems to stem from a failure to distinguish the unit (e.g. one inch) from the entire line segment that is pictured (e.g. 2 1/4 inches). When students are using area models attention need not be focused on the unit; it is understood to be the entire region that is given. Now students must "tune their production" (Anderson, 1983) and refine the conditions, noting that the previous strategy only works for line segments picturing a given unit (e.g. one inch). In other cases, the unit must be discerned before fractional partitions can be assigned.

Behr et al. (1984) also note that generating equivalent fractions may be especially difficult when the lines are partitioned to show an unreduced fraction (2/4) and the child is asked to indicate the reduced form (1/2). The trouble seems to have three sources: (a) identifying the unit (in agreement with Larson), (b) coordinating symbols and pictures, and (c) translating between symbols and pictures, on the number line. The implication is that the amount of information to be stored and compared simultaneously exceeds the limits of processing space.

Discrete (Set-theoretic) Models

Discrete set models can also be used to represent rational numbers using set-subset or set-set relationships. Area models are commonly introduced before discrete set models, as the reverse order may result in interference (Payne, 1976) (confusions about the unit). When part-whole fraction relations are modeled using partitioned areas, the unit is implicit in the whole region and equality of the parts is highlighted. Using sets, the unit is the

total collection of objects in the entire set, with individual elements or partitioned subsets used to designate fractional parts. While the individual elements of the set need not be identical (the same size), textbooks commonly use sets containing identical elements, to introduce this model.

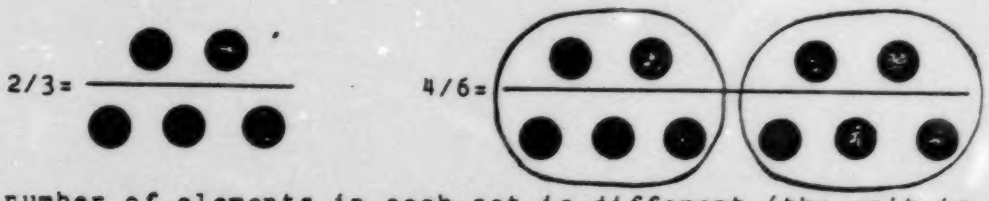
1/2

1/3

During instruction, however, students may suggest set models for consideration in which the elements are not identical (they vary in size, shape, or value). This may cause some confusion for children. (One textbook suggests this subtle distinction need not be discussed with students.) If children note the difference how do they understand and integrate it with what they have previously learned? Teachers will need to monitor students' responses to such situations and deal with confusions or questions that arise (not assume that children will not notice the difference). Perhaps there is a higher order classification level encompassing the elements as equal members, though they remain unequal in other respects (e.g. miniature poodles and Irish setters are members of the class of dogs). Alternatively, the teacher might discuss these examples as illustrations of a different kind of fraction relationship, in which comparisons of numerosity (how many) rather than quantity (how much) are made.

As noted above, area models can be used to generate equivalent fractions without changing the size of the unit. (See the area models of equivalent fractions above.) For set models, equivalent fractions denote a constant relationship between

quantities that are not equivalent (Greeno, 1976). For example:



The total number of elements in each set is different (the unit is 3 and 6 respectively), as is the number of elements in each subset (2 and 4), it is the relationship between the subset (numerator) and the set (denominator) that is constant. Rational number operations which require understanding of invariant relationships, such as multiplication and division, use and benefit from this representation (Greeno, 1976).

The discrete model is often used to find a fractional part of a number, as in "1/2 of 6" below:

Once students become facile at partitioning sets, they are usually introduced to the symbolic algorithm for finding part of a set as follows:

1/2 of 6 means 6 ÷ 2, which equal 3.

Given sufficient set partitioning experiences students could attach meaning to this algorithm, by mapping the actions of set partitioning to the steps of the algorithm. Connections between the partitioning activity and the division computation could also be made by noting what Lawler (1981) terms a "confluence of results". (Partitioning a set of 6 elements into halves results in

subsets of 3 elements each. Dividing 6 by 2 (6 ÷ 2) yields 3 also.) The images formed in partitioning sets could be used to justify the steps of the algorithm and the result obtained if necessary.

Ratios can also be depicted using the discrete model, as in the ratio of 5 apples to 3 oranges. The relationship depicted now, however, involves the comparison of sets (not set and subset), which may be totally disparate. For example, the ratio of people to pieces of pizza might be expressed as 1:2, or "1 to 2", or 1/2, meaning that each person gets two pieces. Note that the conditions under which the discrete model can be used must again be differentiated to include numerical relations between disjoint sets. As Siegler (1982) points out these ratio relationships are very difficult to construct as unitary objects.

Confusions may result because the same symbol "5/3" is used to denote the ratio "five to three" and the fraction "five thirds", while they are read differently. Students are asked to distinguish between the two interpretations based on the context of the problem. Old rules for naming and interpreting fractions must be differentiated from the new rules, with their retrieval depending upon the mathematical situation encountered.

Students may lack a "mature understanding" of rational number because they fail to see how these forms of rational number are simultaneously distinct and yet linked to each other and the overall concept of rational number (Behr, Lesh, & Post, 1981). Failure to make useful connections between the interpretations of rational number may result when each is learned as an isolated system of concepts and procedures, associated with a particular

mathematical context (a constrained set of environmental cues), which must be duplicated for their retrieval. For example, students might only retrieve procedures for decimal manipulation whenever a problem involves decimals (0.25 X 36) and never consider more efficient solution strategies. (That is, 1/4 = 0.25. Therefore, 0.25 X 36 is the same as 1/4 X 36, which is the same as 36/4, or 9.) Such an impoverished system of rational number knowledge, dependent upon narrowly specified cues, would lead to inflexible performance, with a limited range of application.

Conclusions

A mechanistic approach to rational number learning can have disastrous effects for future learning. Students may not have ways to validate the correct sequence of steps when confusions about order arise. They may be unable to infer missing steps when gaps in memory occur. With no conceptual glue to hold them together, the steps of one procedure may be grafted onto or intrude upon another. When concepts and procedures exist as sterile pockets of knowledge, they do not enrich and support each other.

The integration of and mappings between rules (procedures) and concrete models for fractions could lead to both computational skill and understanding, respectively. Void of understanding, the steps of an algorithm are mere symbol manipulations, a set of mechanically applied rules for getting an answer. On the other hand, algorithmic skills are most appropriate for complex computation. We should not expect that students will need to derive the steps of an algorithm with each application; this would

be very inefficient. However, a rich store of experiences and images should be available to students when necessary for verifying procedures or testing the reasonableness of an answer.

Teachers will need to point out to students the ways in which each rational number interpretation is different from the others and yet what attributes they share. Otherwise students may not see the commonalities that exist. Procedures and concepts may exist in isolation as encapsulated bits of knowledge which rarely are accessed and soon fade from disuse.

The need for instruction which leads to useful connections between conceptual and procedural knowledge has been noted. These connections can be made at the three sites outlined by Hiebert (1984). Research which continues to document the effects of various instructional treatments upon learning and to describe student performance in the domain of rational numbers will be important to teachers, math educators and cognitive psychologists alike. In this way we may be able to design instruction which leads to both skilled performance and conceptual competence (Greeno, Riley, & Gelman, 1984).

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